UNIVERSITY OF PENNSYLVANIA

CYCLICLY CONNECTED CONTINUOUS CURVES WHOSE COMPLEMENTARY DOMAIN BOUNDARIES ARE HOMEOMORPHIC, PRESERVING BRANCH POINTS

VIRGIL W. ADKISSON

A THESIS

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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O krzywych cyklicznie spójnych, których obszary uzupełnienia mają brzegi homeomorficzne z zachowaniem punktów rozgałęzienia.

Komunikat przedstawiony przez K. Kuratowskiego dn. 11 października 1930 r.

Przedmiotem badań jest klasyfikacja i własności krzywych, wymienionych w tytule.

V. W. Adkisson.

Cyclicly connected continuous curves whose complementary domain boundaries are homeomorphic, preserving branch points.

Mémoire présenté par M. C. Kuratowski dans la séance de 11 Octobre 1930.

Introduction. It is the purpose of this paper to classify certain cyclicly connected continuous curves [1]) in Euclidian space of two dimensions.

The curves M considered [2]), among others, are those that contain only a finite number of simple closed curves [3]), and are such that any two complementary domain boundaries are homeomorphic, preserving branch points [4]). Such curves are classified according to the number and orders ot the branch points [5]) on each complementary domain boundary.

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[1]) Cf. G. T. Whyburn, *Cyclicly Connected Continuous Curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31—38.

[2]) See *Notation and Definitions* in Part II.

[3]) For a characterization of such curves see, P. Alexandroff, *Über endlich-hoch zusammenhängende stetige Kurven*, Fundamenta Mathematicae, vol. 13 (1929), pp. 34—41.

[4]) Two complementary domain boundaries, J and J', will be said to be homeomorphic if a continuous (1—1) correspondence, π, can be set up between them such that if A is a branch point on J of order α, then $\pi(A) = A'$ is a branch point on J' of order α.

[5]) Cf. Menger's definition of order of a point on a curve, K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1927), pp. 279—280. A branch point, as here used, will refer to a point of Menger order greater than two.

In Part I is established a necessary and sufficient condition that a continuous (1—1) correspondence [1]) of M (on a sphere S) into itself be extendable to S [2]). Part II is devoted to the classification of the curves M indicated above. In Part III it is proved that any continuous (1—1) correspondence between any two complementary domain boundaries of M (on a sphere S) can be extended to M, and also to S, for the curves in which each complementary domain boundary contains exactly three branch points. Furthermore, it is shown that for any one of the curves M any continuous (1—1) correspondence sending the curve into itself can be extended to S.

Credit is due Professor J. R. Kline whose suggestions led to the present paper, and whose constant advice and criticisms have aided materially in its completion.

PART I.

Definition [3]). If M is a cyclicly connected continuous curve lying on a sphere S, π a continuous (1—1) correspondence such that $\pi(M)=M$, C and C' any two simple closed curves of M such that $\pi(C)=C'$, $S-C=R_1+R_2$, and $N_i=M\cdot R_i$, we shall say that regions are preserved under π provided that it is possible to so assign subscripts to the regions of $S-C'$, R_1' and R_2', so that $\pi(N_i)=N_i'$, where $N_i'=M\cdot R_i'$.

Theorem 1. *If M is a cyclicly connected continuous curve lying on a sphere S, and π is a continuous* (1—1) *correspondence such that $\pi(M)=M$, a necessary and sufficient condition that π be extendable to S is that regions be preserved under π.*

Proof: The condition is necessary. For suppose there were a simple closed curve, C of M, enclosing N_1 of M in one region R_1, and N_2 of M in the other region R_2, such that $\pi(C)=C'$

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[1]) The (1—1) correspondences considered in this paper are all (1—1) reciprocal.

[2]) If π is a continuous (1—1) correspondence such that $\pi(M)=M$, then π is extendable to S provided there exists a continuous (1—1) correspondence T such that $T(S)=S$, and such that for points of M, $T=\pi$. See H. M. Gehman, *On Extending a Continuous* (1—1) *Correspondence of Two Plane Continuous Curves to a Correspondence of Their Planes,* Transactions of the American Mathematical Society, vol. 28 (1926), pp. 252—265.

[3]) Cf. H. M. Gehman, loc. cit. Definition, p. 260.

does not enclose $\pi(N_1) = N_1'$ in one region R_1', and $\pi(N_2) = N_2$, in the other region R_2'. (Either N_1 or N_2 may be vacuous. If both vacuous, the necessity of the theorem is trivial).

There are two cases to consider.

Case 1. If N_1 vacuous, then under our assumption N_2' must be divided into two sets, N_{21}' and N_{22}', such that N_{21}' lies in R_1' and N_{22}' lies in R_2'. Let t be a simple arc joining a point of $\pi^{-1}(N_{21}') = N_{21}$ to a point of N_{22} and lying entirely in R_2. Then $\pi(t) = t'$ must join a point of N_{21}' to a point of N_{22}' if π is to be extendable. Hence, t' must have a point P' in common with C'. But $\pi^{-1}(C') = C$, hence P must lie on C. But since P' lies on t', P must lie on t, and since t has no points in common with C, we have a contradiction.

Case 2. If neither N_1 nor N_2 is vacuous, and regions not preserved under π, there must be points of N_1' and N_2' lying together in one of the regions R_i' $(i = 1, 2)$. Let P_1' be a point of N_1' and P_2' a point of N_2' lying together in the region R_1'. Let t' be a simple arc joining P_1' to P_2', and lying entirely within R_1'. Then $\pi^{-1}(t') = t$ is an arc joining $\pi^{-1}(P_1') = P_1$ to $\pi^{-1}(P_2') = P_2$. But P_1 must lie in N_1, and P_2 in N_2. Hence, any arc joining P_1 and P_2 must have points on C. But this would mean that the point P on C corresponds under π^{-1} to two points, which is impossible if π is (1—1).

The condition is sufficient. Let C be any simple closed curve of M. Now consider one of the regions R_i $(i = 1, 2)$ bounded by C and the points N_i of M lying in R_i. Since regions are preserved under π, $\pi(N_1) = N_1'$ lies in the region R_1' bounded by C', and no other points of M lie in this region. By making use of a theorem due to Gehman [1]), we see that the correspondence can be extended to the regions R_1 and R_1'. In order to do this we let R_2 and R_2' correspond to the unbounded domains in Gehman's Theorem. Thus any simple closed curve lying in R_1 (or in R_1') will have an interior and exterior such that N_2 (or N_2') lies in its exterior. Now since regions are preserved under π, any simple closed curve, J of M, lying in $R_1 + C$, and enclosing the points N_{11} of N_1 in its interior, must go into $\pi(J) = J'$ enclosing the points N_{11}' of N_1' in its interior.

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1) Cf. H. M. Gehman, loc. cit., Theorem II, p. 261.

For if this were not the case there would be points of N_{11}' lying outside J', that is, lying in the exterior of J' with N_2' and C'. But this would mean that regions are not preserved under π. Hence, for the regions R_1 and R_1', „interiors are preserved" and we can apply Gehman's Theorem showing that the correspondence can be extended to these two regions. But precisely the same argument holds for R_2 and R_2', and thus for the whole sphere S.

Theorem 2. *If M is a cyclicly connected continuous curve lying on a sphere S, and π is a continuous (1—1) correspondence such that $\pi(M) = M$, a necessary and sufficient condition that π be extendable to S is that for every boundary, J, of a complementary domain of M, $\pi(J) = J'$ be also the boundary of a complementary domain of M.*

Proof: The condition is necessary, for if J is the boundary of a complementary domain and J' fails to be the boundary of a complementary domain then obviously regions are not preserved under π. (Cf. Th. 1).

The condition is sufficient. For suppose that under the hypothesis of this theorem regions are not preserved. Then there is a simple closed curve, C of M, enclosing the points N_1 of M in a region R_1, and the points N_2 of M in a region R_2, such that C' does not enclose N_1' in a region R_1' and N_2' in a region R_2'. Then either N_1 is vacuous while N_2' lies in both R_1' and R_2', or points of N_1' and N_2' lie together in R_1' or R_2'. Suppose N_1 vacuous. This means that C is the boundary of a complementary domain, and under our hypothesis C' must be the boundary of a complementary domain. But this contradicts the assumption on C and C'.

Now suppose neither N_1 nor N_2 vacuous, and points of N_1' lie in one of the regions R_i' with points of N_2'. There are two cases to consider.

Case 1. Suppose all of N_1' and N_2' lie together in one of the regions R_i', say R_2'. Then C' must be the boundary of the complementary domain R_1'. But this implies that C is also the boundary of a complementary domain, contrary to assumption.

Case 2. Suppose a part of N_1', say N_{11}', lies in R_2' with points of N_2'. This means that N_1', and hence N_1, must be disconnected. For if N_1 were connected, there would be at least

one arc of N_1' joining points of N_{11}' in R_2' to points of N_{12}' in R_1' (where $N_1' = N_{11}' + N_{12}'$). This arc would necessarily have at least one point, P', in common with C', which means that for some point P_1 on C, $\pi(P_1) = P'$, and for some point P_2 in N_1, $\pi(P_2) = P'$, and thus π would not be (1—1).

Now consider the complementary domain in R_1 bounded by points of N_{11}, C and N_{12}. That such a domain exists may be shown as follows. Let P_1 be any point in N_{11}, and P_2 any point in N_{12}. Let d be an arc joining P_1 and P_2, but lying entirely within C. On the arc d from P_1 to P_2, let Q_1 be the last point d has in common with N_{11}. Such a point exists since $N_{11} \cdot C$ is closed. On the arc from P_1 to P_2 let Q_2, after Q_1, be the first point d has on N_{12}. Since $N_{11} \cdot N_{12} = 0$, $Q_1 \neq Q_2$. Therefore, the subarc $d' = \underset{\frown}{Q_1 Q_2}$ of d must lie in a complementary domain of M. Suppose the boundary J of this domain contained no points of C. Then $J = J_1 + J_2$ (where $J_1 \subset N_{11}$ and $J_2 \subset N_{12}$) would necessarily be disconnected, which is impossible since J must be a simple closed curve. Hence, there are points of C, N_{11} and N_{12} on J. But any simple closed curve J composed of points of C, N_{11} and N_{12} must correspond under π to a simple closed curve J' composed of points of C', N_{11}' and N_{12}'. But such a simple closed curve cannot bound a complementary domain, for N_{11}' lies in R_2' and N_{12}' lies in R_1', and hence, J' would divide S into two regions, each containing points of C'. Therefore, we have J the boundary of a complementary domain while J' is not the boundary of a complementary domain, contradicting our hypothesis in the theorem.

Theorem 3. *If M is a cyclicly connected continuous curve lying on a sphere S, such that no two complementary domain boundaries of M have a disconnected set in common, then every continuous* (1—1) *correspondence π such that $\pi(M) = M$ can be extended to S.*

Proof: Assume that under this hypothesis π is not extendable. Then from Theorem 2, we see there must be at least one simple closed curve J which is the boundary of a complementary domain, R_1 of M, such that $\pi(J) = J'$ is not the boundary of a complementary domain. If $M = J$, or J plus a simple arc t having only two points in common with J, it is easy to see, by making use of Theorem 2, that Theorem 3 is true. Therefore,

let us assume that M consists of points other than J and t. Let $M = J + N$, and since J is the boundary of a complementary domain R_1, N must lie entirely in the region R_2 bounded by J. Since J' is not the boundary of a complementary domain, $\pi(N) = N'$ must lie a part in R_1' and a part in R_2'. Let $N' = N_1' + N_2'$ such that N_1' lies in R_1', and N_2' lies in R_2'. Then N must be disconnected in such a manner that neither $N_1 = \pi^{-1}(N_1')$ nor $N_2 = \pi^{-1}(N_2')$ contains a limit point of the other. For suppose N_1 contained a limit point P of N_2. Then $\pi(P) = P'$ must lie in R_1', while N_2' lies entirely in R_2'. Hence, P' cannot be a limit point of N_2'. But since P is a limit point of N_2, if the correspondence were continuous P' would be a limit point of N_2'. Thus we have a contradiction, and N must be disconnected in the manner indicated.

Now let P_i be the limit points of N_1 on J, and Q_i the limit points of N_2 on J. Since $M = J + N_1 + N_2$ is connected, J must contain at least one limit point, P_1, of N_1 and one limit point, Q_1, of N_2. For obviously N_1 and N_2 cannot contain limit points of the closed set J. (It is possible that $P_1 = Q_1$). Now P_1 and Q_1 are not the only limit points of N lying on J. For if they were M would not be cyclicly connected [1]). Since P_1 is a limit point of N_1, there must be at least one simple arc from P_1 to some other point P_2 on J such that this arc is a subset of $N_1 + P_1 + P_2$. In precisely the same manner we see there must be an arc $Q_1 TQ_2$ from Q_1 to some other point Q_2 on J such that this arc is a subset of $N_2 + Q_1 + Q_2$.

Let M_i be a maximal connected subset of N_i, and let $P_1 ZP_2$ be that particular arc of $M_1 + P_1 + P_2$ which can be taken with one of the arcs of J from P_1 to P_2, say $P_1 X_1 P_2$, to form the boundary of the region of R_2 containing $M_1 - \underbrace{P_1 ZP_2}$. Now M_2 lies either in the region of R_2 bounded by $P_1 X_2 P_2 ZP_1$, or $P_1 X_1 P_2 ZP_1$, where $J = P_1 X_1 P_2 X_2 P_1$. There are two cases to consider.

Case 1. Suppose M_2 lies in the region of R_2 bounded by $P_1 X_2 P_2 ZP_1$. On the arc $P_1 X_2 P_2$ there must be at least two points of Q_i since M_2 lies inside the region of R_2 bounded by $P_1 X_2 P_2 ZP_1$. From the point P_1 in the order of $P_1 X_2 P_2$ on J

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[1]) Cf. G. T. Whyburn, loc. cit., Theorem I, p. 31.

there must be a first limit point, T_1, of M_2. (It may be that $T_1 = P_1$). From the point P_2 in the order $P_2 X_2 P_1$ on J there must be a first limit point, T_2, of M_2. (It may be that $T_2 = P_2$ but $T_2 \neq T_1$, for this would mean only one point of Q_i on the arc $P_1 X_2 P_2$). Now there must exist a complementary domain, D of M, lying entirely in R_2 whose boundary contains as a proper subset the arc $P_1 Z P_2$, the arc $P_1 T_1$ of J taken in the order $P_1 X_2 P_2$, and the arc $P_2 T_2$ of J taken in the order $P_2 X_2 P_1$. It is easy to see that this boundary cannot contain the whole arc $P_1 X_2 P_2$ of J. For if it did the simple closed curve $P_1 X_2 P_2 Z P_1$ would enclose points of M_2 in a region of R_2, and hence would not be the boundary of a domain D. Therefore, the points on the boundary of D that are also on J must form a disconnected set. But J is itself the boundary of a complementary domain R_1. Hence, we have two complementary domains, R_1 and D, with a disconnected set in common contrary to our hypothesis, and the assumption that π be not extendable cannot hold in this case.

Case 2. Now suppose M_2 lies in the region of R_2 bounded by $P_1 X_1 P_2 Z P_1$, i. e., in the same region with M_1. From P_1 in the order $P_1 X_1 P_2$ on J there must be, just as in case 1, a first limit point T_1 of M_2. From P_2 on J in the order $P_2 X_1 P_1$ there must be a first limit point T_2 of M_2. ($T_1 \neq T_2$). Now from T_1 to T_2 there must be an arc $T_1 V T_2$ which is a subset of $M_2 + T_1 + T_2$ such that if $T_2 X_3 T_1$ is an arc on J, $T_1 V T_2 X_3 T_1$ encloses a region of R_2 containing $M_2 - T_1 V T_2$. Now from T_1 on J in the order $T_1 X_2 T_2$ there must be a first limit point, U_1, of M_1, where U_1 may be T_1. From T_2 on J in the order $T_2 X_2 T_1$ there must be a first limit point, U_2, of M_1. $U_2 \neq U_1$. The arc $U_2 T_2 V T_1 U_1$ must be a proper subset of the boundary of a complementary domain D lying entirely in R_2. But this leads to the same contradiction as in case 1. Therefore, the theorem must be true.

PART II.

Notation and definitions. The symbol M will designate a cyclicly connected continuous curve containing a finite number of simple closed curves [1]), and hence, a finite number of comple-

[1]) If M contains only a finite number of simple closed curves, it follows that M contains no domain.

mentary domains D_i with boundaries J_i, such that for any i and j, J_i is homeomorphic with J_j. The curve M may lie either on the sphere or in the plane. It is convenient to consider M in the plane when speaking of interiors and exteriors.

The orders of the branch points A, B, C, D, E will be denoted by α, β, γ, δ, ε respectively.

The number of branch points on J_i will be denoted by k.

If F_1 is any complementary domain boundary of M, let F_2 be a complementary domain boundary that has an arc $A_1 X_1 B_1$ on F_1. If P_1 is the first branch point on F_2 from B_1 in the order $A_1 X_1 B_1 Y_1 P_1$, let F_3 be the boundary [1]) that contains the arc $B_1 Y_1 P_1$ of F_2. Let B_2 be the last point that F_3 has in common with F_1 such that on F_1 we have the order $A_1 X_1 B_1 B_2$. (It may happen that $B_2 = B_1$). Let P_2 be the first branch point on F_3 from B_2 in the order $P_1 Y_1 B_1 B_2 Y_2 P_2$. Let F_4 be the boundary of M that contains the arc $B_2 Y_2 P_2$. In this manner we define F_i for all boundaries that have points on F_1. Let C_i be the outer boundary of $F_1 + F_2 + \cdots + F_i$, and R the number of boundaries with points on F_1. Then if $F_i (i = 2, 3 \ldots R)$ does not have a disconnected set on C_{i-1}, C_R is a simple closed curve, and we can define $F_i (i = R+1, \ldots)$ for those boundaries that have points on C_R in the same manner as $F_i (i = 2, 3 \ldots R)$ was defined. Let $(S - R)$ be the number of boundaries on C_R. If $F_i (i = R+1, \cdots S)$ does not have a disconnected set on C_{i-1}, C_S will be a simple closed curve, and we can define C_T as C_R and C_S were defined. This process can be continued so long as the curve does not end or F_i does not have a disconnected set on C_{i-1}.

Theorem 4. *If each complementary domain boundary of M contains only two branch points, then M consists of b distinct arcs $(b > 2)$ connecting the branch points A and B.*

Proof: Let J be the boundary of any complementary domain of M, A and B the two branch points of J, and M_1 the points A and B plus all simple arcs of M that join A to B. These arcs can have no branch points other than A and B, otherwise there would exist a boundary with at least three branch points.

[1]) Since M contains no domain, the term *boundary* when used with reference to M will mean complementary domain boundary.

Now suppose M consists of points other than M_1. Let P be such a point. The points P and A must lie on a simple closed curve of M; also the points P and B. Hence, there are two independent arcs from P to A, PXA and PYA. One of these arcs may pass through B. If it does we then have a simple arc from A to B passing through P, and P must belong to M_1, contrary to assumption. Let PZB be an arc from P to B lying on the simple closed curve that contains P and B such that, Q, the last point this arc has in common with $PXAYA$ is different from A. We now have an arc from A to B passing through the branch point Q. But Q is different from A and B, and there cannot be an arc from A to B having branch points other than A and B. Therefore, $M = M_1$.

Theorem 5. *This theorem will consist of several parts,* 5.1, 5.2, 5.3 *and* 5.4.

5.1. *If M is such that $k > 5$, then F_i does not have a disconnected set on C_{i-1}. (If $i = 1$, C_0 is to be taken as the boundary of any complementary domain with points on F_1).*

Proof. Suppose M lies in the plane and two boundaries, J and J', have a disconnected set in common such that no two boundaries lying within the outer boundary C of $J + J'$ have a disconnected set in common. If M contains boundaries that have disconnected sets in common this choice of J and J' is always possible. Otherwise M would necessarily contain an infinite number of complementary domains. Let $K_n = AXBYA$ be the simple closed curve lying within C such that $AXB \subset J$, and $AYB \subset J'$. Let A_1 be the first branch point from A on AYA_1B. There must be at least one such branch point different from B, otherwise there would lie inside K_n a boundary with a disconnected set on J. Let D_1 be the complementary domain lying inside K_n and having the arc AYA_1 on its boundary. Let D_2 be the complementary domain within K_n that lies adjacent to D_1 and has at least one boundary point on AXB, D_3 the complementary domain that lies adjacent to D_2 and has at least one boundary point on AXB, etc. Let J_i be the boundary of D_i. Then $J_1 \neq K_n$, for otherwise we would have a boundary with only two branch points, A and B. Let K_i be the outer boundary of $J_1 + J_2 + \cdots J_i$. Now J_2 cannot have a disconnected set on J_1 because of the choice of J and J'. However, J_i may have

a disconnected set on $K_{i-1}+K_n$, or a disconnected set on K_{i-1} if $i=3, 4\ldots$ Let J_R be the last boundary that has points on AXB, and such that its domain lies inside K_n.

First, suppose that $J_i\,(i=2, 3\ldots R)$ does not have a disconnected set on $K_{i-1}+K_n$, or on K_{i-1}. Then $K_R \neq K_n$. For consider any boundary J_i. At most J_i has two branch points on AXB and therefore, has at least four branch points on $A_1X_1B_1$; where $A_1X_1B_1 = K_R - \underset{\frown}{A_1AXBB_1}$. Since only two of these four branch points can be points of complementary domains lying inside K_R and adjacent to J_i, two of them (different from A and B) must have arcs extending into the exterior of K_R. Therefore, $K_R \neq K_n$.

Now if all branch points on K_n have not been exhausted, we can define $J_i\,(i=R+1, R+2,\ldots.)$ for boundaries with points on $A_1X_1B_1$ as $J_i\,(i=1, 2,\ldots.R)$ was defined for boundaries with points on AXB. Let the number of such boundaries be $(S-R)$. Then $K_S \neq K_n$. For at most, $J_i\,(i=R+1,\cdots S)$ can have only three branch points on K_R. If it had four, only two of these could lie on boundaries that lie inside K_S and adjacent to J_i, while the other two must lie on a boundary lying between J and K_R, having only these two points on K_R. But such a boundary would then have four branch points on AXB which is impossible. Hence, J_i must have at least three branch points on $A_2X_2B_2$; where $A_2X_2B_2 = K_S - \underset{\frown}{A_2AXBB_2}$. Only two of these three can belong to complementary domains inside of K_S and adjacent to J_i, and therefore, at least one has arcs going into the exterior of K_S. Hence, $K_S \neq K_n$.

We can form K_T similar to K_R and K_S, and the corresponding $A_3X_3B_3$, etc., until all of the branch points on K_n have been exhausted, or until some J_i is encountered which has a disconnected set on K_{i-1} (or on $K_{i-1}+K_n$). We propose to show that in either case the boundaries continue indefinitely, which means that M must necessarily contain an infinite number of complementary domains.

If J_i does not have a disconnected set on either K_{i-1} or $K_{i-1}+K_n$, suppose the branch points on K_n have been exhausted at A_n, B_n. There are two cases, (1) $A_n \neq B_n$, (2) $A_n = B_n$.

Case (1). Since the branch points of K_n have been exhausted at A_n, B_n, there are no branch points on $\underset{\frown}{A_n y_n B_n} = K_n - A_n AXBB_n$. Let D_n be the complementary domain that has the arc $A_n y_n B_n$ as a subset of its boundary, and lies within K_n. Let $A_n y_n B_n Z_n A_n$ be the boundary of D_n. Now since there are no branch points on $\underset{\frown}{A_n y_n B_n}$, there must be a branch point on $\underset{\frown}{A_n Z_n B_n}$. Moreover, there are branch points on $A_n X_n B_n$ with arcs extending into the exterior of $A_n X_n B_n BXAA_n$ for the same reason that there were branch points on $A_2 X_2 B_2$ with arcs extending into the exterior of K_S. (It is to be noted that there must exist more than one such branch point on $A_n X_n B_n Z_n A_n$, otherwise M would not be cyclicly connected). We can now use the interior of $A_n X_n B_n Z_n A_n$ and the arc $A_n X_n B_n$ in the same manner as the interior of K_n and the arc AXB were used originally to show that the complementary domains of M must continue to extend into the interior of $A_n X_n B_n Z_n A_n$.

Case 2. In this case $A_n X_n B_n$ is a simple closed curve. Let $\bar{A}$ and $\bar{B}$ be any two adjacent branch points of $\underset{\frown}{A_n X_n B_n}$. Let $A X_n B$ be the arc of the simple closed curve $A_n X_n B_n$ that does not contain A_n. We can now use the interior of $A_n X_n B_n$ and the arc $A X_n B$ as the interior of K_n and the arc AXB were used originally, and show, as in case (1), that the complementary domains of M must continue to extend into the interior of $A_n X_n B_n$.

We have now shown that (provided J_i does not have a disconnected set on K_{i-1}) as soon as the branch points on K_n are exhausted, K_n can be replaced by another simple closed curve to which the original argument, used for the interior of K_n, can be applied and so continue indefinitely.

In the second place, suppose J_c has a disconnected set on K_{c-1} (or on $K_{c-1} + K_n$). Let $A_c X_c B_c y_c A_c$ be the simple closed curve that lies inside the outer boundary of $J_c + K_{c-1}$ such that $A_c X_c B_c \subset K_{c-1}$ and $A_c y_c B_c \subset J_c$. (If J_c has a disconnected set on $K_{c-1} + K_n$ we take $A_c X_c B_c y_c A_c$ such that $A_c X_c B_c \subset K_{c-1} + K_n$, and $A_c y_c B_c \subset J_c$. The argument is practically the same in either case).

Now repeat the process inside $A_c X_c B_c Y_c A_c$ that was used for the interior of K_n, etc., until we come to the last time that J_i has a disconnected set on K_{i-1}. Suppose this occurs at $i=b$. By adding primes to the notation originally employed, we have the simple closed curve $K_n' = A'X'B'Y'A'$, such that $A'X'B' \subset K_{b-1}$, $A'Y'B' \subset J_b$, K_R' = the outer boundary of complementary domains that lie inside K_n' and have points on $A'X'B'$, ect.

Any boundary J_i' that lies inside K_n' and has points on $A'X'B'$ can have at most three branch points on $A'X'B'$. For suppose J_i' has four branch points, P_1, P_2, P_3, and P_4 on $A'X'B'$, and situated in the order $A'P_1P_2P_3P_4B'$. The two branch points P_2 and P_3 cannot have arcs extending into the exterior of K_{b-1}, for such arcs would have to extend into the interior of J_i'. Therefore, P_2 and P_3 have arcs extending into the interior of K_{b-1}, and since they are adjacent they must lie on the same boundary, J_H', which lies interior to K_{b-1}. Furthermore, J_H' can have only these two branch points on $A'X'B'$, for otherwise it would have a disconnected set on $A'X'B'$. If K_T is the last simple closed curve of the K's encountered before reaching K_{b-1}, J_H' must have four branch points on K_T which, in turn, means a complementary domain with four branch points on K_S, etc., down to K_R. But this is impossible since no boundary within K_R can have more than two branch points on AXB.

Since J_i' can have at most three branch points on $A'X'B'$, it has at least three on $A_1'X_1'B_1'$, one of which must have arcs extending into the exterior of K_R'. Therefore, $K_R' \neq K_n'$. Now since for no i does J_i, inside K_n', have a disconnected set on K_{i-1}' (or on $K_{i-1}' + K_n'$) we can repeat the argument indefinitely, each time obtaining a K' with branch points extending into its exterior. Therefore, the assumption that two boundaries of M have a disconnected set in common leads to a contradiction of the hypothesis that M have only a finite number of complementary domains.

Now by assuming that F_i has a disconnected set on C_{i-1} a similar contradiction is reached, which requires a repetition of practically the same argument just completed. Therefore, the theorem is true.

5.2. *If $k=3$, and each branch point of M is at least of order 6, then F_i does not have a disconnected set on C_{i-1}.*

Proof: We use the same notation as in 5.1, and assume that no two boundaries, other than J and J', within the outer boundary of $J+J'$ have a disconnected set in common. Suppose J ane J' have the disconnected set H in common. Since $k=3$, H consists of just two points, or an arc and one point. Suppose the latter. There can be no branch point on $\overset{\frown}{AXB}$ or $\overset{\frown}{AYB}$ for this would require more than three branch points on either J or J'. Therefore, K_n is either the boundary of a complementary domain with only two branch points, A and B, which is impossible, or there lie within K_n two boundaries with a disconnected set $(A+B)$ in common, contrary to assumption.

Assume that H consists of just two points, A and B. Since $k=3$, there is only one branch point, A_1, on $\overset{\frown}{AYB}$, and one branch point, P, on $\overset{\frown}{AXB}$. The points A_1 and P must exist, otherwise there would exist within the outer boundary of $J+J'$ two boundaries with a disconnected set in common. If J_i $(i=1, 2....R)$ does not have a disconnected set on K_{i-1}, then $A_1X_1B_1$ $(B_1=A_1)$ is a simple closed curve. Consider one of the complementary domains that has just one boundary point, P, on $\overset{\frown}{AXB}$. Such a domain must have two branch points on $A_1X_1A_1$, and each such branch point must have at least two arcs extending into the interior of $A_1X_1A_1$. But if there are two such branch points on $A_1X_1A_1$ there must be a third. For if not, there would exist within $A_1X_1A_1$ two boundaries with a disconnected set in common. Hence, in this case we can also obtain $A_2X_2B_2$.

Let S_1, S_2, S_3,... be the simple closed curves used in the succeeding steps of the above argument, i. e., $K_n=S_1$, $A_1X_1A_1=S_2$, $A_iX_iB_iyA_i=S_i$. On S_i $(i=1, 2...)$ there must always occur at least one branch point with arcs going into the interior of S_i. For suppose there were no such branch point on S_i. Let P be a branch point on S_i. From P there must extend at least three arcs, each arc ending on $A_{i-1}X_{i-1}B_{i-1}$. Let the three corresponding branch points on $A_{i-1}X_{i-1}B_{i-1}$ be P_{11}, P_{12} and P_{13}, such that the order on the arc is $A_{i-1}X_{i-1}P_{11}P_{12}P_{13}B_{i-1}$. Now since $k=3$, there must extend from P_{12} three arcs into the exterior of S_{i-1}. Let the three corresponding branch points on $A_{i-2}X_{i-2}B_{i-2}$ be P_{21}, P_{22} and P_{23} such that the order on the arc is $A_{i-2}X_{i-2}P_{21}P_{22}P_{23}B_{i-2}$.

There must extend from P_{22} at least three arcs into the exterior of S_{i-2}. Evidently we can continue in this manner until AXB is encountered. But there cannot exist on AXB three branch points such that the one lying between A and B has arcs extending into the exterior of S_1 (interior of J). Therefore, on S_i there must exist at least one branch point with arcs going into the interior of S_i. (This argument also shows that the number of such arcs going into S_i from any branch point must be at least two). But it is shown above that if there is one such branch point on S_i there must be at least three. Therefore, by selecting two adjacent branch points, A_i and B_i, on S_i that have arcs extending into the interior of S_i, and using the method already indicated for obtaining the S's it is possible to get S_{i+1}, then S_{i+2}, etc., indefinitely unless J_i has a disconnected set on K_{i-1}.

The case where J_i has a disconnected set on K_{i-1} is treated in a manner similar to that in 5.1. Therefore, if F_i has a disconnected set on C_{i-1}, M cannot contain a finite number of complementary domains.

5.3. *If $k \geqq 4$, and each branch point is at least of order 4, then F_i cannot have a disconnected set on C_{i-1}.*

Proof: Practically the same argument can be used here as in 5.1 and 5.2, the essential feature being (as above) that each S_i must have branch points with arcs extending into its interior. If there were an S_i with no such branch point, it is shown, as in 5.2, that this would require a branch point on AXB with arcs extending into the interior of J which is impossible.

5.4. *If* (1) $k=3$, $p=3, 4, 5$,
or (2) $k=4$, $p=3$,
or (3) $k=5$, $p=3$,

where p is the order of the branch points of M, then F_i cannot have a disconnected set on C_{i-1}.

Proof: Case (1a). The theorem is trivial here.

Case (1b). $k=3$, $p=4$. Suppose two boundaires, J and J', have a disconnected set H in common. Let J, J', K_n, etc., be chosen as in 5.1. Since $k=3$, H must consist of two points, A and B, or an arc and one point. The latter is impossible (Cf. 5.2). Suppose then, that $H=A+B$. The following possibilities occur.

(a) $\underset{\frown}{AXB}$ with no branch point, and
$\underset{\frown}{AYB}$ „ „ „ „ .

(b) $\underset{\frown}{AXB}$ with one branch point, and
$\underset{\frown}{AYB}$ „ no „ „ .

(c) $\underset{\frown}{AXB}$ with one branch point, and
$\underset{\frown}{AYB}$ „ „ „ „ .

(a) Since A and B are of order 4, K_n will be a complementary domain with only two branch points, which is impossible.

(b) Since $\underset{\frown}{AYB}$ contains no branch point, there must be a boundary lying with in K_n that has a disconnected set on J. But this contradicts the choice of J and J'.

(c) Let P_1 be the branch point on $\underset{\frown}{AXB}$, and P_2 the branch point on $\underset{\frown}{AYB}$. In this case there must exist two boundaries within K_n with the disconnected set P_1+P_2 in common, contrary to assumption.

Now suppose F_i $(i=3, 4\ldots7)$ has a disconnected set on C_{i-1}. If this does not happen before F_1, the curve is complete and the case proved. If H is the disconnected set that F_i has on C_{i-1} then H must consist of just two points, A and B. Now $\underset{\frown}{AXB}$ cannot have more than one branch point. This is easily seen by construction. Furthermore, $\underset{\frown}{AYB}$ cannot have more than one branch point since F_i already has two, i. e., A and B. We now have a repetition of cases (a), (b) and (c).

Case (1c). This case can be treated in very much the same manner as (1b).

Case (2). If J and J' have a disconnected set H in common, then H must consist of exactly two arcs. But this means there can be no branch point on either $\underset{\frown}{AXB}$ or $\underset{\frown}{AYB}$ since J and J' already have four branch points each. Therefore, K_n must be a boundary with only two branch points, which is impossible. By construction it is easy to see that no $F_i(i=3,4,5)$ can have a disconnected set on C_{i-1}.

Case (3). As in case (2), H must consist of two arcs. Therefore, $\underset{\frown}{AXB}$ and $\underset{\frown}{AYB}$ can have at most one branch point

each, and again, we have a repetition of cases (a), (b), and (c). Suppose $F_i (i = 3, 4 \ldots 11)$ has a disconnected set on C_{i-1}. By construction we wee that, at most, $\overset{\frown}{AYB}$ can have one branch point, and when one branch point does occur on $\overset{\frown}{AYB}$ there can be only one on $\overset{\frown}{AXB}$. Hence, cases (a), (b) and (c) again.

Theorem 6. *If M' is a cyclicly connected continuous curve containing no domain, and such that the boundary of each complementary domain contains exactly three branch points of orders α', β' and γ', and if $\alpha' > \beta' > \gamma' > 4$, then M' must contain an infinite number of complementary domains.*

Proof: In the first place, A', B' and C' must all be of even order. For consider the arcs extending from a branch point of order α'. Such arcs must alternate in going to branch points of orders β' and γ', otherwise two branch points of the same order would lie on the same complementary domain boundary. Hence, A' must be of even order. For the same reason, B' and C' must be of even order. Let $\alpha' = 2\alpha$, $\beta' = 2\beta$, and $\gamma' = 2\gamma$.

Now delete each arc of M' that extends from a branch point of order γ' to a branch point of order α' or β'. This gives a curve M for which $k = 2\gamma \geq 6$, and the branch points on each complementary domain boundary of M alternate with orders α and β. Let F_1 be the boundary of a complementary domain of M. Let C_1 be the outer boundary of all domains that have points on F_1. In general, let C_i be the outer boundary of domain boundaries with points on C_{i-1}. Then C_i will be a simple closed curve since F_i cannot have a disconnected set on C_{i-1}, otherwise the theorem would be true by virtue of 5.1. Let t_i be the total number of branch points on C_i. Let m_i be the number of branch points of C_i that have arcs extending into the interior of C_i, and $n_i = t_i - m_i$. Each point of m_i will have only one arc extending into the interior of C_i since each boundary of M has more than three branch points. It will be convenient to call the branch points of order α the A-points, and likewise for the B's. Let a_i be the number of A-points of m_i and b_i the number of B-points of m_i, i. e. $m_i = a_i + b_i$.

Since the A-points and B-points alternate on the domain boundaries, each B-point on F_1 ends in an A-point on $\overline{C}_1$, and each A-point ends in a B-point; it not, the theorem would be true by virtue of 5.1. Since there are γ B-points and γ A-points on F_1, and as there are $(\alpha - 2)$ arcs extending from A into the exterior of F_1 (likewise for B) we have,

$$a_1 = \gamma(\beta - 2), \qquad b_1 = \gamma(\alpha - 2).$$

Now each A-point on $\overline{C}_{i-1}$ ends in a B-point on $\overline{C}_i$, and each B-point ends in an A-point. But each A-point of m_{i-1} has $(\alpha - 3)$ arcs extending into the exterior of $\overline{C}_{i-1}$ while each A-point of n_{i-1} has $(\alpha - 2)$ such arcs. The same is true of the B's. There are a_{i-1} A-points in m_{i-1}, and $\left(\frac{t_{i-1}}{2} - a_{i-1}\right)$ A-points in n_{i-1}. The expression $\left(\frac{t_{i-1}}{2} - a_{i-1}\right)$ is obtained by noting that the A's and B's alternate on any $\overline{C}_i$ and hence, $\frac{t_{i-1}}{2}$ gives the total number of each in $\overline{C}_{i-1}$.

As each arc extending from $\overline{C}_{i-1}$ to $\overline{C}_i$ gives rise to a point of m, on $\overline{C}_i$, we can write

$$m_i = a_{i-1}(\alpha - 3) + \left(\frac{t_{i-1}}{2} - a_{i-1}\right)(\alpha - 2) +$$

$$b_{i-1}(\beta - 3) + \left(\frac{t_{i-1}}{2} - b_{i-1}\right)(\beta - 2).$$

In order to compute n_i we need to distinguish between the domain boundaries that have an arc in common with $\overline{C}_{i-1}$, and those that have only one point. We designate them by D-boundaries and R-boundaries respectively. Each D boundary gives rise to $(2\gamma - 4)$ points in n_i, and each R-boundary $(2\gamma - 3)$ points in n_i. Now for each A-point in m_{i-1} we have $(\alpha - 4)$ R-boundaries, or $(\alpha - 4)$ $(2\gamma - 3)$ points in n_i, and since there are a_{i-1} such points, the total contribution to n_i from this source is a_{i-1} $(\alpha - 4)$ $(2\gamma - 3)$. For each A-point in n_{i-1} we have $(\alpha - 3)$ R-boundaries, or $(\alpha - 3)$ $(2\gamma - 3)$ points in n_i, and since

there are $\left(\frac{t_{i-1}}{2} - a_{i-1}\right)$ such points, the total contribution to n from this source is $\left(\frac{t_{i-1}}{2} - a_{i-1}\right)(\alpha - 3)(2\gamma - 3)$.

The B-points in m_{i-1} and n_{i-1} give similar expressions by changing the α's and a's to β's and b's.

As there are t_{i-1} branch points on $\overline{C}_{i-1}$, there are t_{i-1} D-boundaries. The number of points in n_i arising from this source is $t_{i-1}(2\gamma - 4)$.

We now have for the final total of n_i,

$$n_i = a_{i-1}(\alpha - 4)(2\gamma - 3) + \left(\frac{t_{i-1}}{2} - a_{i-1}\right)(\alpha - 3)(2\gamma - 3) +$$
$$b_{i-1}(\beta - 4)(2\gamma - 3) + \left(\frac{t_{i-1}}{2} - b_{i-1}\right)(\beta - 3)(2\gamma - 3) +$$
$$t_{i-1}(2\gamma - 4).$$

Let $d_i = n_i - m_i$.

Then $d_i = Gm_{i-1} Hn_{i-1}$, where

$G = (\gamma - 2)(\alpha + \beta) - 6\gamma + 11$, and

$H = (\gamma - 2)(\alpha + \beta) - 4\gamma + 7$.

If d_i is positive, the branch points of M continue to increase as i increases, and M will necessarily contain an infinite number of complementary domains. Since $H - G = 2\gamma - 4 > 0$, we see that d_i is certainly positive when $G > 0$, i. e., when

$$\gamma > 2 + \frac{1}{\alpha + \beta - 6}.$$

But in our hypothesis we have $2\alpha > 2\beta > 2\gamma > 4$. Therefore, the above condition is always satisfied, and the theorem is true.

Classification of the curves M. If any one of the curves M is such that each branch point is of order p, Theorem 5 shows that F_i cannot have a disconnected set on C_{i-1}. Therefore, we can use an argument similar to that used by Benton [1]) to show that M in this case exists with a finite number of complementary domains only when

[1]) Cf. T. C. Benton, *On Continuous Curves Which are Homogeneous Except for a Finite Number of Points,* Fundamenta Mathematicae, vol. 13 (1929), pp. 172—174, section 7.

[A] (1) $k=3,\quad p=3,\ 4,\ 5$
(2) $k=4,\quad p=3$
(3) $k=5,\quad p=3.$

We now consider all curves M for which $k=3$, and in which the branch points are not necessarily all of the same order. In the proof of Theorem 8 it will be shown that these curves are all unique.

(1) $\alpha=\beta=\gamma=3$, where α, β, γ represent the orders of the branch points on each complementary domain boundary. In light of Theorem 5.4, a curve of this type exists, i. e., the curve which is homeomorphic with the edges and vertices of a regular tetrahedron.

(2) $\alpha=\beta=3,\quad \gamma>3.$

This curve does not exist. Let the three arcs extending from a point, A, of order 3 be AB_1, AB_2 and AB_3; where B_1, B_2 and B_3 are the first branch points occuring on the arcs from A. Suppose $\beta_1=\beta$. Then $\beta_2=\gamma$, for B_1, A and B_2 all lie on the boundary of the same complementary domain. But for the same reason $\beta_3=\gamma$, and since B_3, A and B_2 all lie on the same boundary, $\beta_3=\beta$. This is impossible since $\beta\neq\gamma$.

(3) $\alpha=3,\quad \beta=\gamma=2n\quad (n=2,\ 3\ldots.)$

Suppose this curve M exists. Then remove every arc that extends from a point of order 3 to a point of order $2n$. For example, take a branch point A of order 3. The three arcs from A extend to points of order $2n$. The arcs going from a point B of order $2n$ alternate in going first to a point of order 3, then to one of order $2n$. Otherwise we would have two branch points of order 3 on the same boundary. Therefore, when the indicated arcs have been removed we have a curve $\bar{M}$, each boundary of which contains exactly 3 branch points of order n. But $\bar{M}$, and hence M, can exist only when $n<6$ (Cf. [A]).

(3.1) $\alpha=3,\quad \beta=\gamma=4.$

This curve is obtained as follows. Construct on the sphere a curve M which is homeomorphic with the edges of a regular

tetrahedron. Then triangulate [1]) one complementary domain of M, that is, select a point P in a complementary domain and draw from this point three mutually exclusive arcs to the branch points lying on the domain boundary.

(3.2) $\alpha = 3, \quad \beta = \gamma = 6.$

For this curve triangulate each complementary domain of the curve homeomorphic with the regular tetrahedron.

(3.3) $\alpha = 3, \quad \beta = \gamma = 8.$

Triangulate the regular octahedron.

(3.4) $\alpha = 3, \quad \beta = \gamma = 10.$

Triangulate the regular icosahedron.

(3.5) $\alpha = 3, \quad \beta = \gamma = 2n + 1.$

This curve does not exist. For the arcs extending from a point, B, of order $(2n+1)$ must alternate in going to points of order α and γ; otherwise there would exist a boundary with two branch points of order α, or three of order γ.

(3.6) $\alpha = 3, \quad 3 < \beta \neq \gamma > 3.$

This curve does not exist for the same reason that the curve in (2) does not exist.

Now consider those curves ($k = 3$) in which at least one branch point of every boundary is of order 4.

(4) $\alpha = \beta = \gamma = 4.$

This curve is homeomorphic with the regular octahedron.

(5) $\alpha = \beta = 4, \quad \gamma > 4.$

Construct a regular pyramid with a base of γ vertices. Then triangulate the base.

(6) $\alpha = 4, \quad \beta = \gamma = 2n.$

In this curve delete all arcs extending from points of order 4. This gives a curve $\bar{M}$ for which $k = 4$, and $p = n$. But $\bar{M}$, and hence M, can exist only when $n < 4$, (Cf. [A]).

(6.1) $\alpha = 4, \quad \beta = \gamma = 6.$

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[1]) This is a term adopted from. O. Veblen, *American Mathematical Society Colloquium Lectures*, vol. 5, Part II, p. 41.

Triangulate the cube.

(6.2) $\alpha = 4, \quad \beta = \gamma = 2n + 1.$

This curve does not exist for the same reason that the curve in (2) does not exist.

(7) $\alpha = 4, \quad \beta > \gamma > 4.$

In this case B and C must each be of even order, (Cf. 2). Therefore, let $\beta = 2m$ and $\gamma = 2n$. Delete all arcs extending from points of order 4. This gives a curve $\bar{M}$ for which $k = 4$, and the branch points on each boundary are alternately of orders m and n. Now subdivide each complementary domain of $\bar{M}$ by an arc extending between the two points of order m. We then have a curve M' for which $k = 3$, $\alpha' = \beta' = 2m$, and $\gamma' = n$. In M' delete each arc which extends from a point of order n, obtaining a curve $\bar{M}'$ for which $k = n$, and $p = m$. Therefore, $\bar{M}'$, and hence M, exists only when $n = 3$, $m = 3, 4, 5$; $n = 4$, $m = 3$; $n = 5$, $m = 3$. (Cf. [A]).

(7.1) $\alpha = 4, \quad \beta = 6, \quad \gamma = 8.$

This curve may be constructed as follows. Subdivide each complementary domain of the curve (3.3) by extending an arc from the branch point of order 3 to any point, P, lying between the two points of order 8. The arc joining these two points of order 8 is common to two domains, and hence, P is used in the subdivision of each of these domains. Therefore, P becomes a branch point of order 4, while each branch point of order 3 is doubled.

(7.2) $\alpha = 4, \quad \beta = 6 \quad \gamma = 10.$

This curve is obtained from the curve of (3.4) as (7.1) was obtained from (3.3).

(8) Now consider the cases that have not already been disposed of for which $\alpha = \beta \neq \gamma$. Both A and B must be of even order, Cf. (2). Let $\alpha = \beta = 2n$, and $\gamma = m$. Delete the arcs from the points of order γ, and obtain a curve for which $k = \gamma$, and each branch point is of order n. (Cf. [A]).

(8.1) $\alpha = \beta = 6, \quad \gamma = 5.$

Triangulate a regular dodecahedron. Theorem 6 shows that all curves M for which $k=3$, and $\alpha>\beta>\gamma>4$ cannot exist with a finite number of complementary domains. This completes the classification of the curves when $k=3$.

Before considering the curves for which $k=4$, we prove the following theorem.

Theorem 7. *If M' is a cyclicly connected continuous curve, containing no domain, such that,* (1) *every complementary domain boundary has only a finite number of branch points,* (2) *any two boundaries are homeomorphic,* (3) *F_i does not have a disconnected set on C_{i-1},* (4) *M' has an infinite number of complementary domains; and if $\overline{M}$ is a cyclicly connected continuous curve such that,* (1) *any two complementary domain boundaries of $\overline{M}$ are homeomorphic,* (2) *any complementary domain boundary of $\overline{M}$ is homeomorphic with any complementary domain boundary of M' except for one branch point, which is of higher order in $\overline{M}$ than in M', then $\overline{M}$ also contains an infinite number of complementary domains.*

Proof. Let F_i, C_i and $\overline{F}_i$, $\overline{C}_i$ be defined for M' and $\overline{M}$ respectively, as F_i and C_i were originally defined for M. (Cf. *Notation and Definitions*).

Since M' contains an infinite number of complementary domains, there must be arcs extending into the exterior of C_R. Hence, there must be arcs extending into the exterior of $\overline{C}_R$. For since all of the branch points on F_1 are of the same order as those on F_1, except one which is of higher order, there must lie between F_1 and $\overline{C}_R$ at least one more complementary domain than lies between F_1 and C_R of M'. Furthermore, since no F_i can have a disconnected set on C_{i-1}, and no $\overline{F}_i$ a disconnected set on $\overline{C}_{i-1}$ (otherwise the theorem would be true by virtue of Th. 5) this extra domain requires that the number of arcs extending into the exterior of $\overline{C}_R$ be greater than the number extending into the exterior of C_R. But this, in turn, means a greater number of complementary domains lying between $\overline{C}_R$ and $\overline{C}_S$ than lie between C_R and C_S of M'. And since there are arcs of M' extending into the exterior of C_S, there must be

arcs of $\overline{M}$ extending into the exterior of $\overline{C}_S$. But this continues indefinitely for M', and therefore for $\overline{M}$. Hence, $\overline{M}$ must contain an infinite number of complementary domains. This completes the proof.

We now investigate the cases for which $k=4$.

(9) $\alpha=\beta=\gamma=\delta=3$.

This curve exists, and is homeomorphic with the edges of a cube.

(10) $\alpha=\beta=\gamma=3, \quad \delta>3$.

Such a curve may be obtained by subdividing the two complementary domains, D_1 and D_2, of a simple closed curve as follows. Select a point, P_1, in D_1. From P_1 draw δ mutually exclusive arcs to δ distinct points, Q_1, $Q_2, \ldots Q_\delta$, on C. Now select δ points, R_1, $R_2, \ldots R_\delta$, on C so that the first branch point on either side of an R-point is a Q-point. Then from a point, P_2, in D_2 draw δ arcs to these δ R-points. This gives the required curve.

(11) $\alpha=\beta=3, \quad \gamma=\delta>3$.

Such a curve is impossible if A and B lie adjacent. For consider a branch point, A, of order 3 that lies adjacent to D. There are three arcs from A, and they must alternate in going to points of order 3 and order δ. But since A is of odd order, this is impossible. Hence, assume that A and B separate C and D on the complementary domain boundary. Divide each complementary domain by an arc joining the two points of order $\gamma=\delta$. This gives a curve $\overline{M}$ for which $k=3$, $\overline{\alpha}=3$, $\overline{\gamma}=\overline{\delta}=2\gamma$. But it has been shown that $\overline{M}$ exists only when $\overline{\gamma}=\overline{\delta}=8, 10$. Cf. (3). Therefore, M exists only when $\gamma=\delta=4, 5$.

(12) $\alpha=3, \quad \beta=\gamma=\delta=4$.

Subdivide the complementary domains of the curve $\overline{M}$ for which $\overline{k}=3$, $\overline{\alpha}=\overline{\beta}=\overline{\gamma}=4$. Let D_1 be any complementary domain of $\overline{M}$. Let P be any point in D_1. From P draw three arcs to three distinct points on the boundary of D_1 in such a manner that any two are separated by two of the three branch points $\overline{A}$, $\overline{B}$ and $\overline{C}$. Every other complementary domain of $\overline{M}$ is

subdivided in the same manner, inserting only one branch point between any two branch points of $\overline{M}$.

(13) $\alpha = 3, \quad \beta = \gamma = \delta \geqq 5.$

Delete all arcs of this curve that extend from points of order 3. This gives a curve $\bar{M}$ for which $k = 6$, and each branch point is of order ≥ 3. But since we have shown that the curve for which $k = 6$ and $p = 3$ does not exist with a finite number of complementary domains (Cf. [*A*]), we can make use of Theorem 7 to show that the curve in this case does not exist.

(14) $\alpha = 3, \quad \beta = \gamma \neq \alpha, \quad \alpha \neq \delta \neq \beta.$

A and D must separate B and C since A is of odd order. Divide each complementary domain by an arc joining the two branch points A and D. This gives a curve M for which $k = 3$, $\bar{\alpha} = 2\alpha$, $\bar{\delta} = 2\delta$ and $\bar{\beta} = \beta$. The point B must be of even order. If $\bar{\delta} \leq 10$ and $\bar{\beta} = 4$, $\bar{M}$ exists, and therefore M exists.

(15) We have shown that for the special case $k = 4$, and all branch points are of order 4, M must have an infinite number of complementary domains. Therefore, from this fact, and Theorem 7, we conclude that no curve M exists when $k = 4$, and the branch points of lowest order are $\geqq 4$.

This completes the classification of the curves M when $k = 4$. Now we consider the curves for which $k = 5$.

(16) $\alpha = \beta = \gamma = \delta = \varepsilon = 3.$

This curve exists, and is homeomorphic with the regular dodecahedron.

(17) $\alpha = \beta = \gamma = \delta = 3, \quad \varepsilon > 3.$

If we delete all arcs extending from points of order ε, we obtain a curve $\bar{M}$ for which $k = \varepsilon$, and $p = 3$. But $\bar{M}$, and hence M, exists only when $\varepsilon \leqq 5$.

(18) $\alpha = \beta = \gamma = 3, \quad \delta > 3, \quad \varepsilon > 3.$

Suppose D and E do not lie adjacent on the domain boundary. Delete all arcs going from points of order δ. This gives a curve $\bar{M}$ for which $k = \delta$ and $p = \varepsilon$. But such a curve cannot exist with a finite number of complementary domains (Cf. [*A*]). Suppose D

and E lie adjacent. In this case, both D and E must be of even order. Subdivide each complementary domain of M by an arc extending from a point, P, between D and E to the point of order 3 that lies opposite to P, that is, the branch point that is not adjacent to either D or E. This gives a curve $\bar{M}$ for which $k=4$ and each branch point is of order ≥ 4. Therefore, M, and hence M, cannot exist. (Cf. Th. 5.3 and [A]).

(19) $\alpha=\beta=3, \quad \gamma, \delta, \varepsilon>3.$

Since the curve for which $\alpha=\beta=\gamma=3, \quad \delta, \varepsilon>3$ does not exist, Theorem 7 shows that M under the above conditions cannot exist.

(20) $\alpha=3, \quad \beta, \gamma, \delta, \varepsilon>3.$

Since the curve in (19) does not exist, Theorem 7 shows that this curve cannot exist.

(21) Since the curve for which $k=5$ and $p=4$ does not exist, Theorem 7 shows that the curves M for which $k=5$, and the branch points of lowest order are ≥ 4 cannot exist.

This completes the case for $k=5$.

Now consider all curves M for which $k\geq 6$. Since we have shown that these curves do not exist when $p\geq 3$, Theorem 7 shows that no curve for which $k\geq 6$ can exist with a finite number of complementary domains. This completes the classification of the curves M.

PART III.

Theorem 8. *The curves M for which $k=3$ are unique; that is, a curve in which the orders of the branch points on each complementary domain boundary are α, β, γ is homeomorphic with every other curve satisfying the same conditions.*

Proof: Case 1. From Theorem 5.4 it follows that if M lies on a sphere, and $k=3$, $p=3, 4, 5$, then M is homeomorphic with the edges of a regular tetrahedron, octahedron or icosahedron respectively. Therefore, for curves of this type the theorem must be true.

Case 2. $\alpha=3, \quad \beta=\gamma=2n\geq 4.$

Consider the arcs of M that go from a point of order 3. Each such arc will end in a point of order $2n$. These three points of order $2n$ must all lie on a simple closed curve, C, that does not pass through the branch point A. For there must be an arc between any two of these branch points that does not pass through A or the other point of order $2n$; otherwise there would exist a boundary with 3 points of order $2n$ and one of order 3. Delete all arcs extending from points of order 3, giving a curve M. Then C becomes the boundary of a complementary domain of $\overline{M}$. The arcs extending from a point of order $2n$ alternate in going to points of order 3 and $2n$. Therefore, every point of order $2n$ will become a point of order n in $\overline{M}$. Hence, for $\overline{M}$ we have $k=3$, $\overline{\alpha}=\overline{\beta}=\overline{\gamma}=n$.

If $n=2$, $\overline{M}$ is a simple closed curve. For certainly C of M exists. Suppose there were a point P of $\overline{M}$ not on C. Since P is in M, there must exist an arc in M that joins P to C. As the 3 branch points on C are of order 4, there is only one arc from each extending into the exterior of C. These three arcs all go to the same branch point of order 3. For each one of these arcs is common to the boundary of two complementary domains, and each domain can have only one branch point of order 3. Now since A is of order 3, no other arcs can extend from A, and since M is connected, P must lie on one of the 3 arcs going to C from A. Therefore, P cannot lie in $\overline{M}$, and $\overline{M}$ must be a simple closed curve.

For $n=3, 4, 5$ we use the same argument as for $n=2$, and obtain curves $\overline{M}$ for which $k=3$ and $\overline{\alpha}=\overline{\beta}=\overline{\gamma}=3, 4, 5$ respectively.

This shows that the curves considered here must be homeomorphic with the curves obtained by triangulating each complementary domain of the following curves: $k=3$, $p=3, 4, 5$, or by properly subdividing the two complementary domains of a simple closed curve.

Case 3. $\alpha=4, 5$, $\beta=\gamma=6$.

Consider the arcs that go from a point of order 4 (or 5). Each such arc will end in a point of order 6. These 4 (or 5) points of order 6 must lie on a simple closed curve (Cf. Case 1). If

we delete all arcs of M that extend from a point of order 4 (or 5) to one of order 6, we obtain a curve $\overline{M}$ for which $k=4$ (or 5) and $p=3$. This shows that the curves in the case are homeomorphic with the curves obtained by triangulating the curves for which $k=4$ (or 5) and $p=3$.

Case 4. $\alpha \geqq 5, \quad \beta=\gamma=4.$

Delete all arcs extending from points of order α. This gives a simple closed curve as in case 1.

Case 5. $\alpha=4, \quad \beta=6, \quad \gamma=8, 10.$

Delete all arcs extending from points of order 4. This gives a curve $\overline{M}$ for which $k=4$, and the branch points on each boundary are alternately of orders 3 and 4 (or 5). Now if we subdivide each complementary domain of $\overline{M}$ by an arc joining the two points of order 3, we obtain a curve M' for which $k=3$, and $\alpha'=4$ (or 5), $\beta'=\gamma'=6$. But we have shown that the theorem holds for this curve (Cf. case 3). It follows that the theorem must be true for the curve M. This completes the proof for the curves in which $k=3$.

We now consider the uniqueness of the curves for which $k=4$.

(9′) $\alpha=\beta=\gamma=\delta=3.$

The uniqueness of this curve follows from Th. 5. 4.

(10′) $\alpha=\beta=\gamma=3, \quad \delta>3.$

Delete all arcs extending from a point of order δ. This gives a simple closed curve showing that M in this case is unique.

(11′) $\alpha=\beta=3, \quad \gamma=\delta=4, 5.$

In each complementary domain of M join the two boundary points of order $\gamma=\delta$. This gives a curve $\overline{M}$ for which $k=3$, $\alpha=\overline{\beta}=8$, 10 and $\overline{\gamma}=3$. But $\overline{M}$, and hence M, is unique. (Cf. case 2, Th. 8).

(12′) $\alpha=\beta=\gamma=4, \quad \delta=3.$

This curve is not unique. For if all the arcs extending from points of order 3 be deleted, in one case we obtain a curve $\overline{M}$ for which $k=3$ and $p=4$, while in the other case a curve M' is obtained for which $k=4$, $\alpha'=\beta'=\gamma'=3$ and $\delta'=4$.

(14′) $\alpha = \beta = 4, \quad \gamma = 3, \quad \delta = 5.$

In each complementary domain join the two boundary points of order γ and δ. This gives a curve for which $\bar{\alpha} = 4$, $\bar{\beta} = 6$, $\bar{\gamma} = 10$. Hence M is unique.

The three curves for the case $k = 5$ are each unique.

(16′) $\alpha = \beta = \gamma = \delta = \varepsilon = 3.$

The uniqueness follows from Th. 5.4.

(17′) $\alpha = \beta = \gamma = \delta = 3, \quad \varepsilon = 4, 5.$

Delete all arcs extending from points of order ε. This gives a curve for which $k = 4$, 5 and $p = 3$. Hence, M must be unique.

Theorem 9. *If,* (1) M *lies on a sphere* S, *and* $k = 3$, (2) F_1 *and* F_1' *are any two complementary domain boundaries of* M, *and* π *a continuous* (1—1) *correspondence such that* $\pi(F_1) = F_1'$ [1]), *then there exists a continuous* (1—1) *correspondence* T *such that* $T(M) = M$, $T(S) = S$ *and for* F_1 *and* F_1', $T = \pi$.

Proof: Let F_2 be any complementary domain boundary of M with an arc G_1 on F_1. Let F_2' be the boundary with points $G_1' = \pi(G_1)$ on F_1'. For F_2 let T be any continuous (1—1) correspondence such that $T(F_2) = F_2'$, and for G_1 and G_1', $T = \pi$. Let F_3 be a boundary of M that has the arc G_2 on F_2, and has points on F_1. Let F_3' be the boundary of M that has the points $G_2' = \pi(G_2)$ on C_2'. (C_i = the outer boundary of $F_1 + F_2 + \cdots F_i$). Then for F_3, T is any correspondence such that $T(F_3) = F_3'$ and at the same time corresponds to the correspondence already defined for G_2 and G_2'. Since no F_i can have a disconnected set on C_{i-1} (Cf. Theorems 5 and 8) we can continue and define T for any F_i, and therefore, put all of M into correspondence with itself. Hence, $T(M) = M$.

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[1]) The correspondence is such that branch points on F_1 go into branch points on F_1', i. e., if A is a branch point on F_1 of order α, then $\pi(A) = A'$ is a point on F_1', of order α.

Since no two complementary domain boundaries of M can have a disconnected set in common, T can be extended to S (Cf. Th. 3). Therefore, $T(S) = S$.

Theorem 9 holds for the curves M when $k = 4$, excepting case (12′) where $\alpha = \beta = \gamma = 4$, $\delta = 3$. If case (12′) be omitted, practically the same proof as that in Theorem 9 can be given here. However, it can be shown that the Theorem holds for one of the curves in ease (12′), i. e., the curve obtained by properly subdividing the complementary domains of the regular octahedron, Cf. (12). This may be shown as follows. Let M be the regular octahedron and M' the derived curve. Let D and D' be any two complementary domains of M'. Then D will lie in a complementary domain E of M and D' in a complementary domain E' of M. (It may happen that $E = E'$). Any continuous (1—1) correspondence of the boundary J of D into the boundary J' of D' (Cf. footnote to Th. 9) throws an arc t of the boundary F of E into an arc t' of the boundary F' of E'. The arc t (also t') contains one branch point of M of order 4. Let the three points on F that are branch points of M' but not of M be P_1, P_2, P_3, and the corresponding points on F' be P_1', P_2', P_3'. The correspondence can be extended to F and F' such that $\pi(P_i) = P_i'$. $(i = 1, 2, 3)$. Furthermore, the correspondence can be extended to E and E' of M [1]). But we have shown that such a correspondence can be extended to M and also to S. Therefore, the correspondence can be extended to M' and S.

Theorem 9 does not hold for the second curve under case (12′), i. e., the curve derived from the curve in which $\alpha = \beta = \gamma = 3$, $\delta = 4$. Even the extra condition that the correspondence preserve sense (or alter sense) [2]) on each complementary domain boundary of the curve does not suffice for this case.

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[1]) Cf. A. Schoenflies, *Beiträge zur Theorie der Punktmengen,* Mathematische Annalen, vol. 62 (1906), pp. 286—328. See also J. R. Kline, *A New Proof of a Theorem Due to Schoenflies,* Proceedings of the National Academy of Sciences, vol. 6 (1920), pp. 529—531.

[2]) See, J. R. Kline, *A Definition of Sense on Closed Curves in Non-Metrical Plane Analysis Situs,* Annals of Mathematics, vol. 19 (1918), p. 185. In an unpublished paper I have shown that a continuous (1—1) correspondence π such that $\pi(M) = M$, and $\pi(S) = S$ must preserve sense or alter sense, on every complementary domain boundary of M.

In order for Theorem 9 to hold in the case where $k = 5$, we need to add the further condition that the correspondence preserve sense on the complementary domain boundaries of the curve.

Finally, from Theorem 3, we see that any continuous (1—1) correspondence of any one of the curves M ($k = 3, 4, 5$) into itself can be extended to the sphere.

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